7.3 Notes: Linear Systems and Row Operations

A matrix derived from a system of linear equations (each written in standard form with the constant term on the right) is called the Augmented matrix of the system. In addition, the matrix derived from the coefficients of the system (but not including the constant terms) is called the Coefficient matrix.

System: $\begin{cases} x - 4y + 3z = 5 \\ -x + 3y - z = -3 \\ 2x \qquad - 4z = 6 \end{cases}$

$2x - 4z = 6$

Augmented Matrix: $\begin{array}{c} \\ E1 \\ E2 \\ E3 \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & -4 & 3 & 5 \\ -1 & 3 & -1 & -3 \\ 2 & 0 & -4 & 6 \end{array}\right] \end{array}$

Coefficient Matrix: $\begin{bmatrix} 1 & -4 & 3 \\ -1 & 3 & -1 \\ 2 & 0 & -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 6 \end{bmatrix}$

MATRIX EQN $\Rightarrow$ Coeff. M $\cdot$ Variable M = Constant M

Example 1: Write the augmented matrix for the system of linear equations.

$\begin{cases} x + 3y - 3w = 9 \\ -y + 4z + 2w = -2 \\ x - 5z - 6w = 0 \\ 2x + 4y - 3z = 4 \end{cases}$

$\begin{array}{c} \\ E1 \\ E2 \\ E3 \\ E4 \end{array} \begin{array}{c} \begin{array}{ccccc} x & y & z & w & c \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & -3 & 9 \\ 0 & -1 & 4 & 2 & -2 \\ 1 & 0 & -5 & -6 & 0 \\ 2 & 4 & -3 & 0 & 4 \end{bmatrix} \end{array}$

What is the order of the augmented matrix? 4×5

Elementary Row Operations

Operation	Notation
1. Interchange two rows.	$R_a \leftrightarrow R_b$
2. Multiply a row by a nonzero constant.	$cR_a \quad (c \neq 0)$
3. Add a multiple of a row to another row.	$cR_a + R_b$

Example 2: Perform the following row operations on the given matrix.

Note the elementary row operation beside the row that is changed.

a. Interchange the first and second rows of the original matrix.

Original Matrix

$$\begin{bmatrix} 0 & 1 & 3 & 4 \\ -1 & 2 & 0 & 3 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

New Row-Equivalent Matrix

$$R_1 \leftrightarrow R_2 \begin{bmatrix} -1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 2 & -3 & 4 & 1 \end{bmatrix}$$

b. Multiply the first row of the original matrix by $\frac{1}{2}$.

Original Matrix

$$\begin{bmatrix} 2 & -4 & 6 & -2 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\tfrac{1}{2}R_1 \begin{bmatrix} 1 & -2 & 3 & -1 \\ 1 & 3 & -3 & 0 \\ 5 & -2 & 1 & 2 \end{bmatrix}$$

c. Add −2 times the first row of the original matrix to the third row.

Original Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 2 & 1 & 5 & -2 \end{bmatrix}$$

$-2 \quad -4 \quad 8 \quad -6$

New Row-Equivalent Matrix

$$\begin{bmatrix} 1 & 2 & -4 & 3 \\ 0 & 3 & -2 & -1 \\ 0 & -3 & 13 & -8 \end{bmatrix}$$

$R_3 = -2R_1 + R_3$

Example 3: Identify the elementary row operation being performed to obtain the new row-equivalent matrix.

Original Matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & 7 \\ 2 & -6 & 14 \end{bmatrix}$$

New Row-Equivalent Matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 2 & -6 & 14 \end{bmatrix} R_2 = -3R_1 + R_2$$

Example 4: Solve the linear system using its associated augmented matrix and the following row operations.

Linear System

$$\begin{cases} x - 2y + 3z = 9 \\ -x + 3y = -4 \\ 2x - 5y + 5z = 17 \end{cases}$$

Associated Augmented Matrix

$$\begin{array}{c|cccc} & x & y & z & c \\ E.1 & 1 & -2 & 3 & \vdots\ 9 \\ E.2 & -1 & 3 & 0 & \vdots\ -4 \\ E3 & 2 & -5 & 5 & \vdots\ 17 \end{array}$$

Add the first equation to the second equation

row row

$$R_2 = R_1 + R_2 \quad \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

$$-2 \quad 4 \quad -6 \quad -18$$

Add –2 times the first equation to the third equation.

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$R_3 = -2R_1 + R_3$$

Add the second equations to the third equation

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$R_3 = R_2 + R_3$$

Multiply the third equation by $\frac{1}{2}$.

$$x - 2y + 3z = 9$$

$$y + 3z = 5$$

$$z = 2$$

$$\frac{1}{2} R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

At this point, you can use back-substitution to find *x* and *y*.

$$y + 3(2) = 5$$

$$y + 6 = 5$$

$$y = -1$$

$$x - 2(-1) + 3(2) = 9$$

$$x + 2 + 6 = 9$$

$$x + 8 = 9$$

$$x = 1$$

SolN: (x, y, z)

$$(1, -1, 2)$$

The last matrix in Example 4 is said to be in row echelon form. The term ***echelon*** fers to the stair-step pattern formed by the nonzero entries of the matrix.

Row-Echelon Form and Reduced Row-Echelon Form

A matrix in **row-echelon form** has the following properties. (REF)

1. Any rows consisting entirely of zeros occur at the bottom of the matrix.
2. For each row that does not consist entirely of zeros, the first nonzero entry is 1 (called a **leading 1**).
3. For two successive (nonzero) rows, the leading 1 in the higher row is farther to the left than the leading 1 in the lower row.

A matrix in *row-echelon form* is in **reduced row-echelon form** when every column that has a leading 1 has zeros in every position above and below its leading 1. (RREF)

Note - Row-echelon form of a matrix is not unique while reduced row-echelon form of a matrix is unique.

Example 5: Determine whether each matrix is in row-echelon form. If it is, determine whether the matrix is in reduced row-echelon form.

a. $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ Ref Yes RRef No

b. $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -4 \end{bmatrix}$ Ref No RRef No

c. $\begin{bmatrix} 1 & -5 & 2 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Ref Yes RRef No

d. $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Ref Yes RRef Yes

e. $\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ Ref No RRef No

f. $\begin{bmatrix} 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Ref Yes RRef Yes helon

Example 6: Solve the system $\begin{cases} y+z-2w=-3 \\ x+2y-z=2 \\ 2x+4y+z-3w=-2 \\ x-4y-7z-w=-19 \end{cases}$

$$\begin{array}{c|c} & \begin{matrix} x & y & z & w & c \end{matrix} \\ \begin{matrix} E1 \\ E2 \\ E3 \\ E4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & -2 & -3 \\ 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & 1 & -3 & -2 \\ 1 & -4 & -7 & -1 & -19 \end{bmatrix} \end{array}$$

Steps:

Write the system in an Augmented Matrix

Use a graphing calculator (MATRIX) to find RREF

MATRIX EDIT 4×5
Enter elements
QUIT
MATRIX (2nd $\boxed{x^{-1}}$) MATH RREF
MATRIX [A] (enter) RREF([A])

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \begin{matrix} x=-1 \\ y=2 \\ z=1 \\ w=3 \end{matrix}$$

Soln: $\boxed{(-1, 2, 1, 3)}$ ☆

Example 7: Solve the system $\begin{cases} x-y+2z=4 \\ x+z=6 \\ 2x-3y+5z=4 \\ 3x+2y-z=1 \end{cases}$

$$\begin{array}{c} \begin{matrix} x & y & z & c \end{matrix} \\ \begin{bmatrix} 1 & -1 & 2 & 4 \\ 1 & 0 & 1 & 6 \\ 2 & -3 & 5 & 4 \\ 3 & 2 & -1 & 1 \end{bmatrix} \end{array}$$

$$RREF([A]) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} x=0 \\ y=0 \\ z=0 \\ 0=1 \end{matrix}$$

false ☆

No soln.

Example 8: Solve the system $\begin{cases} x-2y+3z=9 \\ -x+3y=-4 \\ 2x-5y+5z=17 \end{cases}$

$$\begin{array}{c|c} & \begin{matrix} x & y & z & c \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \end{array}$$

$$RREF([A]) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\boxed{(1, -1, 2)}$$

Example 9: A system with an Infinite number of solutions

Solve the system $\begin{cases} 2x + 4y - 2z = 0 \\ 3x + 5y = 1 \end{cases}$

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right]$$

$$\text{RREF} \quad \begin{array}{cccc} x & y & z & c \end{array}$$
$$\left[\begin{array}{cccc} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$x + 5z = 2$ $\boxed{x = 2 - 5z}$

$y - 3z = -1$ $\boxed{y = -1 + 3z}$

Solve x, y in terms of z

Soln

$(2 - 5z, -1 + 3z, z)$

Solving Systems Using Inverse Matrices

> A System of Equations with a Unique Solution
>
> If A is an invertible matrix, then the system of linear equations represented by $AX = B$ has a unique solution given by $X = A^{-1}B$.

Example 10: Use an inverse matrix to solve the system

Write a **Matrix Equation** first:

a) $\begin{cases} x + y + z = 10{,}000 \\ 0.06x + 0.075y + 0.095z = 730 \\ x - 2z = 0 \end{cases}$

$x - 2z = 0$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0.06 & 0.075 & 0.095 \\ 1 & 0 & -2 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 10\,000 \\ 730 \\ 0 \end{bmatrix}}_{B}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \cdot B \qquad \begin{bmatrix} 4000 \\ 4000 \\ 2000 \end{bmatrix}$$

b) You invest $15,000 in AAA-rated bonds. AA-rated bonds, and B-rated bonds and want an annual return of $720 ~~$1040~~. The average yields are 3.5% on AAA bonds, 5% on AA bonds, and 6% on B bonds. You invest twice as much in AAA bonds as in AA bonds. Your investment can be represented as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0.035 & 0.05 & 0.06 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15\,000 \\ 720 \\ 0 \end{bmatrix} \qquad \begin{cases} x + y + z = 15{,}000 \\ 0.035x + 0.05y + 0.06z = 720 \\ x - 2y = 0 \end{cases}$$

Where **x**, **y** and **z** represent the amounts invested the amounts invested in AAA, AA, and B bonds, respectively. How much have you invested in each bond? Use an inverse matrix to solve the system.

$A^{-1} \cdot B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6000 \\ 3000 \\ 6000 \end{bmatrix}$$

AAA $6000

AA $3000

B $6000